This test consists of five relays of six questions each. "$TAFTPQITR$" stands for "The Answer From The Previous Question In The Relay", so if question 3 in a relay references $TAFTPQITR$, that is the answer from question 2 in that relay.

The answers to all parts of all Relays are integers.

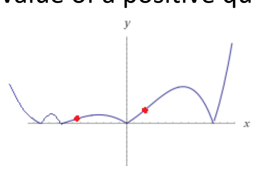
Relay 1

(1) Solution: $\frac{d}{dx}\left[14\pi\sin(\sqrt{x})\right]_{x=\pi^2} = \left[14\pi\cos(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)\right]_{x=\pi^2} = -7$.

(2) Solution: $f'(x) = 2axe^{ax^2} \rightarrow f'(2) = 4ae^{4a}$. Further $f(2) = e^{4a}$ so the equation for the tangent line is $y - e^{4a} = 4ae^{4a}(x-2) \rightarrow y = (4ae^{4a})x - 8ae^{4a} + e^{4a} = (4ae^{4a})x + e^{4a}(1-8a)$. Therefore $TAFTPQITR = -7 = 1 - 8a \rightarrow a = 1$.

(3) Solution: $TAFTPQITR = 1$ so $f(x) = \cfrac{x}{TAFTPQITR + \cfrac{x}{TAFTPQITR + \cfrac{x}{TAFTPQITR + \cfrac{x}{\cdots}}}} \rightarrow y = \frac{x}{1+y} \rightarrow y^2 + y - x = 0 \rightarrow (2y+1)y' = 1 \rightarrow y' = \frac{1}{2y+1}$. When $x = 6$ $y^2 + y - 6 = 0 \rightarrow (y+3)(y-2) = 0 \rightarrow y = 2$ since $f(x) > 0$ when $x > 0$. Therefore $y' = \frac{1}{5}$ and the answer is 5.

(4) Solution: Since $TAFTPQITR = 5$, $f(x) = |3x^4 - 5x^3 - 15x^2 + 25x| = |x(3x^3 - 5x^2 - 15x + 25)| = |x(3x(x^2-5) - 5(x^2-5))| = |x(3x-5)(x^2-5)|$. So this is the absolute value of a positive quartic with four distinct roots, which will look something like:



It will have two points of inflection not at the roots (red dots). This graph has three relative maxima, four relative minima, and changes concavity six times. So the answer is 72.

(5) Solution: The Maclaurin series for $\cos(x^2) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{4k}}{(2k)!}$. By definition, the Maclaurin series takes the form $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}$. Therefore for $n = TAFTPQITR = 72$, $\frac{f^{(72)}(0)x^{72}}{72!} = \frac{(-1)^k x^{4k}}{(2k)!}$. Therefore we are looking at $k = 18$ and so $f^{(72)}(0) = \frac{(72!)}{(36!)}$. Between 37 and 72, {40, 45, 50, 55, 60, 65, 70} have at least one factor of five and there are eight total factors. This means the answer is 8.

(6) Solution: $y' = xy + x - y - 1 \rightarrow y' = (x-1)(y+1) \rightarrow \frac{y'}{y+1} = x - 1 \rightarrow \ln|y+1| = \frac{1}{2}x^2 - x + C \rightarrow y = Ce^{\frac{1}{2}x^2 - x} - 1$. When $= 0$ $Ce^{\frac{1}{2}x^2 - x} - 1 = C - 1 = TAFTPQITR = 8 \rightarrow C = 9$. So

when $x = 4$, $y = 9e^{\frac{1}{2}(16)-4} - 1 = 9e^4 - 1$, so $A = 9$, $B = 4$, and $C = -1$. The answer is therefore -36.

Relay 2

(1) Solution: The answer is a square of side length three, so the area is 9.

(2) Solution: At any water level, the cone will be similar to the cone of the full tank, so $\frac{r}{h} = \frac{6}{100} \rightarrow r = \frac{3}{50}h$. Therefore $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{2500}\right)h^3 \rightarrow \frac{dV}{dt} = \left(\frac{9\pi}{2500}\right)h^2\frac{dh}{dt} \rightarrow \frac{dh}{dt} = \frac{dV}{dt}\frac{2500}{9\pi h^2} = (-TAFTPQITR\pi)\frac{2500}{9\pi(50)^2} = -1$.

(3) Solution: Minimizing the distance will also minimize the distance squared, so it suffices to minimize the distance squared: $Dsq = \left(x - \frac{1}{2}\right)^2 + \left(x^{\frac{3}{2}} - 1 + 1\right)^2 \rightarrow Dsq' = 2\left(x - \frac{1}{2}\right) + 3x^2 = 3x^2 + 2x - 1 = 0 \rightarrow x = \frac{-2\pm\sqrt{4+12}}{6} = -1 \; or \; \frac{1}{3}$. $Dsq'' = 6x + 2$ so at $x = \frac{1}{3}$ Dsq'' is positive and we have a minimum. The minimum distance squared is therefore $Dsq = \left(\frac{1}{3} - \frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{1}{36} + \frac{1}{27} = \frac{7}{108}$. So $\frac{7}{D^2} = 108$.

(4) Solution: Let $a = \sqrt{TAFTPQITR} = \sqrt{108}$ and $b = 6$. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so $\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0 \rightarrow y' = -\frac{xb^2}{ya^2}$. The slope between a point $\left(x, \pm b\sqrt{1 - \frac{x^2}{a^2}}\right)$ on the ellipse and $(12,0)$ is $\frac{\pm b\sqrt{1-\frac{x^2}{a^2}}}{x-12}$ so we are looking for the points at which $\frac{\pm b\sqrt{1-\frac{x^2}{a^2}}}{x-12} = -\frac{xb^2}{\pm ba^2\sqrt{1-\frac{x^2}{a^2}}} \rightarrow b^2a^2\left(1 - \frac{x^2}{a^2}\right) = 12b^2x - b^2x^2 \rightarrow b^2a^2 = 12b^2x \rightarrow x = \frac{a^2}{12} = \frac{108}{12} = 9$. So $y = \pm 6\sqrt{1 - \frac{108}{144}} = \pm\frac{6}{12}\sqrt{144 - 108} = \pm 3$ and the slopes are $= \pm\frac{(9)(36)}{(3)(108)} = \pm 1$. The angle between these lines is 90 degrees.

(5) Solution: The area of that rectangle will be $A = (x)((30)(90) - x^2) = 2700x - x^3 \rightarrow A' = 2700 - 3x^2 = 0 \rightarrow x = 30$ and the area is $A = 30(2700 - 900) = No\ max$.

(6) Unsolvable due to previous incorrect answer. ~~Solution: The amount of salt S in the tank is leaving at $\frac{S}{1000} * 10 = \frac{S}{100}$ mg per minute, so $\frac{dS}{dt} = -\frac{S}{100} \rightarrow \frac{dS}{S} = -\frac{1}{100}dt \rightarrow \ln|S| = -\frac{t}{100} + C \rightarrow S = Ce^{-\frac{t}{100}} \rightarrow S = 54000e^{-\frac{t}{100}} \rightarrow S(200\ln(3)) = 54000e^{-\frac{200\ln(3)}{100}} = 54000e^{\ln\left(\frac{1}{9}\right)} = \frac{54000}{9} = 6{,}000$.~~

Relay 3

(1) Solution: $\int_1^2 (12x^3 - 6x^2 + 8x - 3)dx = [3x^4 - 2x^3 + 4x^2 - 3x]_1^2 = (48 - 16 + 16 - 6) - (3 - 2 + 4 - 3) = 40$.

(2) Solution: $\int_0^1 \frac{x^2+x+1}{2x^3+3x^2+6x+TAFTPQITR} dx = \left[\frac{1}{6}\ln|2x^3 + 3x^2 + 6x + 44|\right]_0^1 = \frac{1}{6}(\ln(55) - \ln(44)) = \frac{1}{6}\ln\left(\frac{5}{4}\right)$. $A + B - C = 7$.

(3) Solution: $3\int_0^{\frac{\pi}{2}} \sin^{TAFTPQITR}(x)\, dx = 3\int_0^{\frac{\pi}{2}} \sin^3(x)\, dx = 3\int_0^{\frac{\pi}{2}} (1 - \cos^2(x))\sin(x) dx = 3\int_0^1 (1 - u^2) du = 3\left[u - \frac{1}{3}u^3\right]_0^1 = 3\left(\frac{2}{3}\right) = 2$.

(4) Solution: $\int_0^{TAFTPQITR} \frac{x^{2018}}{x^{2018}+(TAFTPQITR-x)^{2018}} dx = \int_0^2 \frac{x^{2018}}{x^{2018}+(2-x)^{2018}} dx$. Let $u = 2 - x \to dx = -du$, $u = 2$ when $x = 0$ and $u = 0$ when $x = 2$. So $\int_0^2 \frac{x^{2018}}{x^{2018}+(2-x)^{2018}} dx = \int_0^2 \frac{(2-u)^{2018}}{(2-u)^{2018}+u^{2018}} du \equiv I$. Now $2I = I + I = \int_0^2 \frac{x^{2018}}{x^{2018}+(2-x)^{2018}} dx + \int_0^2 \frac{(2-u)^{2018}}{(2-u)^{2018}+u^{2018}} du = \int_0^2 \frac{x^{2018}+(2-x)^{2018}}{x^{2018}+(2-x)^{2018}} dx = \int_0^2 1 dx = 2$. So $I = 1$.

(5) Solution: $\int_0^{TAFTPQITR} \frac{x^3}{\sqrt{x^4+1}} dx = \int_0^1 \frac{x^3}{\sqrt{x^4+1}} dx$. Let $\tan(u) = x^2$. Then $x^4 + 1 = \sec^2(u)$ and $x dx = \frac{1}{2}\sec^2(x)$. So $\int_0^1 \frac{x^3}{x^4+1} dx = \frac{1}{2}\int_0^{\pi/4} \tan(u)\sec(u)\, du = \frac{1}{2}[\sec(u)]_0^{\pi/4} = \frac{1}{2}(\sqrt{2} - 1) = \frac{-1+\sqrt{2}}{2}$. We know the roots of the polynomial are $\frac{-4\pm\sqrt{16-16K}}{8} = \frac{-1\pm\sqrt{1-K}}{2}$ so therefore $K = -1$.

(6) Solution: $I(b) = \int_1^\infty e^{bx} dx = \frac{1}{b}e^{bx}\Big]_1^\infty = -\frac{e^b}{b}$ for $b < 0$. So $I(TAFTPQITR) = \frac{1}{e}$. So $180e * \frac{1}{e} = 180$.

Relay 4

(1) Solution: $f(x) = g(x) \to 20x^3 - 5x^4 = 0 \to 5x^3(4 - x) = 0 \to x = 0 \text{ or } 4$. Therefore the area is $\int_0^4 20x^3 - 5x^4 dx = [5x^4 - x^5]_0^4 = 4^4(5 - 4) = 256$.

(2) Solution: $\frac{1}{2}\sqrt{TAFTPQITR} = 8$. The two curves intersect when $\sqrt{x} = 8x^2 \to x - 64x^4 = 0 \to x = 0 \text{ or } x = \frac{1}{4}$. Therefore the desired volume is $\pi\int_0^{\frac{1}{4}} x - 64x^4 dx = \pi\left[\frac{1}{2}x^2 - \frac{64}{5}x^5\right]_0^{\frac{1}{4}} = \frac{3}{10}\frac{1}{16}\pi = \frac{3\pi}{160}$. So the answer is 10.

(3) Solution: The desired volume is $2\pi\int_0^1 x^{10}(1 - x)dx = 2\pi\left[\frac{1}{11}x^{11} - \frac{1}{12}x^{12}\right]_0^1 = \frac{2\pi}{132}$. So the answer is 132.

(4) Solution: Let $TAFTPQITR = A$. Then the nonzero intersection point between $y = mx$ and $f(x) = Ax - x^2$ occurs when $mx = Ax - x^2 \to m = A - x \to x = A - m$. The total area of

the region is $\int_0^A Ax - x^2 dx = \left[\frac{1}{2}Ax^2 - \frac{1}{3}x^3\right]_0^A = \frac{A^3}{6}$. Therefore we want $\int_0^{A-m}(A-m)x - x^2 dx = \left[\frac{1}{2}(A-m)x^2 - \frac{1}{3}x^3\right]_0^{A-m} = \frac{(A-m)^3}{6} = \frac{1}{2}\cdot\frac{A^3}{6} \to (A-m)^3 = \frac{A^3}{2} \to m = A - \frac{A}{\sqrt[3]{2}} = A\left(1 - \frac{1}{\sqrt[3]{2}}\right)$. So the answer is $A = TAFTPQITR = 132$.

(5) Solution: Simpson's Rule is exact for cubics, so the answer is just $\int_1^2 4x^3 + 3x^2 + 2x + 132\, dx = [x^4 + x^3 + x^2 + 132x]_1^2 = 16 + 8 + 4 + 264 - 1 - 1 - 1 - 132 = 157$.

(6) Solution: $x^2 - 4x + y^2 + 2y \leq 11 \to (x-2)^2 + (y+1)^2 \leq 16$ is a circle of radius 4 centered at $(2, -1)$. The distance from the center to $3x + 4y + 157 = 0$ is $D = \frac{|3(2)+4(-1)+157|}{\sqrt{3^2+4^2}} = \frac{159}{5}$. Therefore by the Theorem of Pappus the volume is $(16\pi)\left(2\pi\left(\frac{159}{5}\right)\right)$. So $K = 159$.

Relay 5

(1) Solution: $\sum_{n=1}^{\infty}\frac{4n}{3^n} = 4\frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = 3$.

(2) Solution: $\lim_{x\to\infty}\left((x^3+8x^2)^{\frac{1}{3}} - (x^3+2x^2)^{\frac{1}{3}}\right) =$

$$\lim_{x\to\infty}\left(\frac{\left((x^3+8x^2)^{\frac{1}{3}}-(x^3+2x^2)^{\frac{1}{3}}\right)}{1}\frac{\left((x^3+8x^2)^{\frac{2}{3}}+(x^3+8x^2)^{\frac{1}{3}}(x^3+2x^2)^{\frac{1}{3}}+(x^3+2x^2)^{\frac{2}{3}}\right)}{\left((x^3+8x^2)^{\frac{2}{3}}+(x^3+8x^2)^{\frac{1}{3}}(x^3+2x^2)^{\frac{1}{3}}+(x^3+2x^2)^{\frac{2}{3}}\right)}\right) =$$

$$\lim_{x\to\infty}\left(\frac{\left((x^3+8x^2)-(x^3+2x^2)\right)}{(x^3+8x^2)^{\frac{2}{3}}+(x^3+8x^2)^{\frac{1}{3}}(x^3+2x^2)^{\frac{1}{3}}+(x^3+2x^2)^{\frac{2}{3}}}\right) = \lim_{x\to\infty}\left(\frac{(8-2)x^2}{x^2\left(\left(1+\frac{8}{x}\right)^{\frac{2}{3}}+\left(1+\frac{8}{x}\right)^{\frac{1}{3}}\left(1+\frac{2}{x}\right)^{\frac{1}{3}}+\left(1+\frac{2}{x}\right)^{\frac{2}{3}}\right)}\right) = \frac{6}{3} =$$

2.

(3) Solution: $\frac{d}{dx}\int_{x^2}^{x^4}\frac{\sin\left(\frac{\pi}{8}t\right)}{\sqrt{t}}dt = \frac{\sin\left(\frac{\pi}{8}x^4\right)}{x^2}4x^3 - \frac{\sin\left(\frac{\pi}{8}x^2\right)}{x}2x = 4x\sin\left(\frac{\pi}{8}x^4\right) - 2\sin\left(\frac{\pi}{8}x^2\right) = 4(2)\sin\left(\frac{\pi}{8}(16)\right) - 2\sin\left(\frac{\pi}{8}x^2\right) = -2$.

(4) Solution: $\lim_{n\to\infty}\frac{12}{\pi}\sum_{k=1}^{n}\frac{1}{\sqrt{(TAFTPQITR)^2n^2-k^2}} = \frac{12}{\pi}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\sqrt{4-\left(\frac{k}{n}\right)^2}} = \frac{12}{\pi}\int_0^1\frac{1}{\sqrt{4-x^2}}dx = \frac{12}{\pi}\arcsin\left(\frac{1}{2}\right) = \frac{12}{\pi}\frac{\pi}{6} = 2$.

(5) Solution: Using integration by parts, we let $u = \ln(x)$ and $dv = x^{TAFTPQITR}dx = x^2dx \to v = \frac{1}{3}x^3$. So $\int_0^e x^2\ln(x)\,dx = \left[\frac{1}{3}x^3\ln(x)\right]_0^3 - \int_0^e\frac{1}{3}x^3\frac{1}{x}dx = \left[\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3\right]_0^e = \frac{2}{9}e^3$. So the answer is 18.

(6) Solution: Assume $P(x)$ has a root of multiplicity $k > 1$; without loss of generality, we can assume the root occurs at $x = 0$ so that $P(x) = x^k R(x)$ for some $R(x)$ of order $18 - k$ with $R(0) \neq 0$. Then $P''(x) = k(k-1)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x)$. The first term, and the fact that $R(0) \neq 0$, means that x^{k-2} is the highest power of x that is a factor of $P''(x)$. So let $P''(x) = x^{k-2}S(x)$ for some $S(x)$ of order $16 - k + 2 = 18 - k$ with $S(0) \neq 0$. Now $P(x) = q(x)P''(x) \rightarrow x^k R(x) = q(x)x^{k-2}S(x) \rightarrow x^2 R(x) = q(x)S(x)$. Since $R(0) \neq 0$ and $S(0) \neq 0$ that means that the quadratic $q(x)$ must have a root at zero of multiplicity two; i.e., $q(x) = Cx^2$ for some constant C. So $P(x) = Cx^2 P''(x)$. Let $P(x) = \sum_{j=0}^{18} a_j x^j$. Then $P''(x) = \sum_{j=0}^{18} a_j j(j-1)x^{j-2}$. Plugging these in to the previous formula gives $\sum_{j=0}^{18} a_j x^j = \sum_{j=0}^{18} C a_j j(j-1)x^j$. Matching powers gives us $a_j = C a_j j(j-1)$ for all j, which can only be true if $a_j = 0$ when $j < 18$ (you can pick a value of C to make it work for a_{18}. Therefore $P(x) = a_{18}x^{18}$, and it has only one distinct root. So the answer is 1.